

## NEW LOWER BOUNDS FOR THE RANK OF MATRIX MULTIPLICATION

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**ABSTRACT.** The rank of the matrix multiplication operator for  $\mathbf{n} \times \mathbf{n}$  matrices is one of the most studied quantities in algebraic complexity theory. We prove new lower bounds that imply the rank is at least  $3\mathbf{n}^2 - 4\mathbf{n}^{\frac{3}{2}} + \mathbf{n}$ . When  $\mathbf{n}$  is an even square, the bound is  $3\mathbf{n}^2 - 4\mathbf{n}^{\frac{3}{2}} + 3\mathbf{n}$ . The previous lower bound, due to Bläser [1], was  $\frac{5}{2}\mathbf{n}^2 - 3\mathbf{n}$ . The new bounds improve Bläser's bound for all  $\mathbf{n} \geq 37$ . We also prove lower bounds for rectangular matrices and include remarks about the geometry of the matrix multiplication operator that may be of interest in their own right.

## 1. INTRODUCTION

Let  $X = (x_j^i)$ ,  $Y = (y_j^i)$  be  $\mathbf{n} \times \mathbf{n}$ -matrices with indeterminant entries. The *rank* of matrix multiplication, denoted  $\mathbf{R}(M_{(\mathbf{n}, \mathbf{n}, \mathbf{n})})$ , is the smallest number  $r$  of products  $p_\rho = u_\rho(X)v_\rho(Y)$  where  $u_\rho, v_\rho$  are linear forms, such that the entries of the matrix product  $XY$  are contained in the linear span of the  $p_\rho$ . This quantity is also called the bilinear complexity of  $\mathbf{n} \times \mathbf{n}$  matrix multiplication. More generally, one may define the rank  $\mathbf{R}(B)$  of any bilinear map  $B$ , see §2.

From the point of view of geometry, rank is badly behaved as it is not semi-continuous. Geometers usually prefer to work with the *border rank* of matrix multiplication, which fixes the semi-continuity problem by fiat: the border rank of a bilinear map  $B$ , denoted  $\underline{\mathbf{R}}(B)$ , is the smallest  $r$  such that  $B$  can be approximated to arbitrary precision by bilinear maps of rank  $r$ . By definition, one has  $\mathbf{R}(B) \geq \underline{\mathbf{R}}(B)$ .

In [5] G. Ottaviani and I gave new lower bounds for the border rank of matrix multiplication, namely  $\underline{\mathbf{R}}(M_{(\mathbf{n}, \mathbf{n}, \mathbf{n})}) \geq 2\mathbf{n}^2 - \mathbf{n}$ . Those results are used here to prove:

**Theorem 1.1.** *Let  $p \leq \mathbf{n}$  be a natural number. Then*

$$\mathbf{R}(M_{(\mathbf{n}, \mathbf{n}, \mathbf{n})}) \geq (3 - \frac{1}{p+1})\mathbf{n}^2 - (4p+1)\mathbf{n}.$$

*This bound is maximized when  $p = \lfloor \frac{\sqrt{\mathbf{n}}}{2} - 1 \rfloor$  or  $p = \lceil \frac{\sqrt{\mathbf{n}}}{2} - 1 \rceil$ , so if  $\frac{\sqrt{\mathbf{n}}}{2} \in \mathbb{Z}$ ,*

$$\mathbf{R}(M_{(\mathbf{n}, \mathbf{n}, \mathbf{n})}) \geq 3\mathbf{n}^2 - 4\mathbf{n}^{\frac{3}{2}} + 3\mathbf{n}.$$

These bounds extend to partially rectangular matrix multiplication. Let  $M_{(\mathbf{n}, \mathbf{m}, \mathbf{l})}$  denote the multiplication of an  $\mathbf{n} \times \mathbf{m}$  matrix by a  $\mathbf{m} \times \mathbf{l}$  matrix.

**Theorem 1.2.** *Let  $p \leq \mathbf{n}$  be a natural number. Then*

$$\mathbf{R}(M_{(\mathbf{n}, \mathbf{n}, \mathbf{m})}) \geq (1 + \frac{p}{p+1})\mathbf{n}\mathbf{m} + \mathbf{n}^2 - (4p+1)\mathbf{n}.$$

*For example, if  $\frac{\sqrt{\mathbf{n}}}{2} \in \mathbb{Z}$ , taking  $p = \frac{\sqrt{\mathbf{n}}}{2} - 1$  gives*

$$\mathbf{R}(M_{(\mathbf{n}, \mathbf{n}, \mathbf{n})}) \geq 2\mathbf{n}\mathbf{m} + \mathbf{n}^2 - 2\mathbf{n}^{\frac{1}{2}}(\mathbf{m} + \mathbf{n}) + 3\mathbf{n}.$$

The previous bound, due to Bläser [2], was  $\mathbf{R}(M_{(\mathbf{n}, \mathbf{n}, \mathbf{m})}) \geq 2\mathbf{n}\mathbf{m} - \mathbf{m} + 2\mathbf{n} - 2$ .

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*Remark 1.3.* There is no known nontrivial lower bound on rank in terms of border rank for matrix multiplication or tensors in general. Regarding upper bounds, if  $T$  is a tensor of border rank  $r$ , where the approximating curve of rank  $r$  tensors limits in such a way that  $q$  derivatives of the curve are used, then the rank of  $T$  is at most  $(2q - 1)r$ , see [3, Prop. 15.26].

The language of tensors will be used throughout. In §2 the rank question and matrix multiplication are rephrased in the language of tensors. In §3 the equations used to obtain lower bounds for border rank in [5] are expressed in a manner suitable for the proof of Theorem 1.2, which is proved in §4. I work over the complex numbers throughout.

## 2. MATRIX MULTIPLICATION AND ITS RANK

Let  $A, B, C$  be vector spaces, of dimensions  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  and with dual spaces  $A^*, B^*, C^*$ . That is,  $A^*$  is the space of linear maps  $A \rightarrow \mathbb{C}$ . Write  $A^* \otimes B$  for the space of linear maps  $A \rightarrow B$  and  $A^* \otimes B^* \otimes C$  for the space of bilinear maps  $A \times B \rightarrow C$ . To avoid extra  $*$ -s, I work with bilinear maps  $A^* \times B^* \rightarrow C$ , i.e., elements of  $A \otimes B \otimes C$ . Let  $T : A^* \times B^* \rightarrow C$  be a bilinear map. One may also consider  $T$  as a linear map  $T : A^* \rightarrow B \otimes C$  (and similarly with the roles of  $A, B, C$  exchanged), or as a trilinear map  $A^* \times B^* \times C^* \rightarrow \mathbb{C}$ .

The *rank* of a bilinear map  $T : A^* \times B^* \rightarrow C$ , denoted  $\mathbf{R}(T)$ , is the smallest  $r$  such that there exist  $a_1, \dots, a_r \in A$ ,  $b_1, \dots, b_r \in B$ ,  $c_1, \dots, c_r \in C$  such that  $T(\alpha, \beta) = \sum_{i=1}^r a_i(\alpha)b_i(\beta)c_i$  for all  $\alpha \in A^*$  and  $\beta \in B^*$ . Equivalently (see e.g., [4, §3.1]), considering the map  $T : A^* \rightarrow B \otimes C$ ,  $\mathbf{R}(T)$  is the smallest number of rank one elements of  $B \otimes C$  needed to span a linear space containing the linear space  $T(A^*)$ .

Let  $M_{(\mathbf{m}, \mathbf{n}, \mathbf{l})} : \text{Mat}_{\mathbf{m} \times \mathbf{n}} \times \text{Mat}_{\mathbf{n} \times \mathbf{l}} \rightarrow \text{Mat}_{\mathbf{m} \times \mathbf{l}}$  denote the matrix multiplication operator. Write  $M = \mathbb{C}^{\mathbf{m}}$ ,  $N = \mathbb{C}^{\mathbf{n}}$  and  $L = \mathbb{C}^{\mathbf{l}}$ . Then

$$M_{(\mathbf{m}, \mathbf{n}, \mathbf{l})} : (N \otimes L^*) \times (L \otimes M^*) \rightarrow N \otimes M^*$$

has the interpretation as  $M_{(\mathbf{m}, \mathbf{n}, \mathbf{l})} = \text{Id}_N \otimes \text{Id}_M \otimes \text{Id}_L \in (N^* \otimes L) \otimes (L^* \otimes M) \otimes (N \otimes M^*)$ , where  $\text{Id}_N \in N^* \otimes N$  is the identity map. If one thinks of  $M_{(\mathbf{m}, \mathbf{n}, \mathbf{l})}$  as a trilinear map  $(N \otimes L^*) \times (L \otimes M^*) \times (N \otimes M^*) \rightarrow \mathbb{C}$ , in bases it is  $(X, Y, Z) \mapsto \text{trace}(XYZ)$ . If one thinks of  $M_{(\mathbf{m}, \mathbf{n}, \mathbf{l})}$  as a linear map  $N \otimes L^* \rightarrow (L^* \otimes M) \otimes (N \otimes M^*)$  it is just the identity map tensored with  $\text{Id}_M$ . In particular, if  $\alpha \in N \otimes L^*$  is of rank  $q$ , its image, considered as a linear map  $L \otimes M^* \rightarrow N \otimes M^*$ , is of rank  $q\mathbf{m}$ .

Returning to general tensors  $T \in A \otimes B \otimes C$ , from now on assume  $\mathbf{b} = \mathbf{c}$ . When  $T = M_{(\mathbf{m}, \mathbf{n}, \mathbf{l})}$ , one has  $A = N^* \otimes L$ ,  $B = L^* \otimes M$ ,  $C = N \otimes M^*$ , so  $\mathbf{b} = \mathbf{c}$  is equivalent to  $\mathbf{l} = \mathbf{n}$ . For any tensor  $T \in A \otimes B \otimes C$ , if there exists  $\alpha \in A^*$  such that  $T(\alpha)$  is of maximal rank  $\mathbf{b}$ , then one may use the linear map  $T(\alpha) : B^* \rightarrow C$  to identify  $C \simeq B^*$ , and consider  $T(A^*) \subset B \otimes B^*$  as a subspace of the space of linear maps  $B \rightarrow B$ .

Applying this to matrix multiplication with  $\mathbf{l} = \mathbf{n}$ ,  $M_{(\mathbf{n}, \mathbf{n}, \mathbf{m})} \in (N^* \otimes L) \otimes (L^* \otimes M) \otimes (N \otimes M^*)$ , a choice of  $\alpha^0$  also allows one to identify  $L \simeq N$ , which, letting  $\mathfrak{gl}(N)$  denote the algebra of linear maps  $N \rightarrow N$ , induces a map

$$(1) \quad M_A : \mathfrak{gl}(N) \rightarrow \mathfrak{gl}(B).$$

It follows immediately that:

**Key Observation 2.1.** The map (1) is an inclusion of Lie algebras.

That the rank of a tensor can be studied via a subspace of endomorphisms was the idea behind Strassen's equations for border rank [7]. He observed that if  $\mathbf{a} = 3$ , and one takes a basis  $\alpha^0, \alpha^1, \alpha^2$  of  $A^*$  such that  $T(\alpha^0)$  has maximal rank  $\mathbf{b}$  (this implies  $\mathbf{R}(T) \geq \mathbf{b}$ ), then  $\mathbf{R}(T) = \mathbf{b}$  if and only if  $T(\alpha^1), T(\alpha^2)$ , considered as endomorphisms, commute. More generally, he showed

that the border rank is bounded below by  $\mathbf{b}$  plus half the rank of their commutator considered as a linear map,  $[T(\alpha^1), T(\alpha^2)] : B \rightarrow B$ . To see the first assertion, note that if  $T(\alpha^1), T(\alpha^2)$  commute and are diagonalizable, then they are simultaneously diagonalizable, and each is a linear combination of the  $\mathbf{b}$  rank one elements on the diagonal.

It follows from these remarks that Strassen's commutator  $[M(\alpha^1), M(\alpha^2)]$  of  $\mathbf{b} \times \mathbf{b}$  matrices (where  $\mathbf{b} = \mathbf{n}\mathbf{m}$ ), has rank equal to  $\mathbf{m}$  times the rank of the commutator of  $\mathbf{n} \times \mathbf{n}$  matrices  $[\alpha^1, \alpha^2]$ . In particular, for generic  $\alpha^1, \alpha^2$ , it will be of maximal rank. In general, the Key Observation allows one to transport questions about expressions in commutators in  $\mathfrak{gl}(B) = \mathfrak{gl}_{\mathbf{n}^2}$  to expressions in commutators in  $\mathfrak{gl}(N) = \mathfrak{gl}_{\mathbf{n}}$ .

### 3. THE EQUATIONS OF [5] IN COORDINATES

Strassen's equations were rephrased in [6] as follows: given  $T \in A \otimes B \otimes C$ , consider  $T \otimes Id_A \in A \otimes B \otimes C \otimes A \otimes A^* = A^* \otimes B \otimes A \otimes A \otimes C$ , and its skew-symmetrization in the  $A \otimes A$  factor,  $T_A^{\wedge 1} \in A^* \otimes B \otimes \Lambda^2 A \otimes C$ , which we think of as a linear map  $A \otimes B^* \rightarrow \Lambda^2 A \otimes C$ . If  $\mathbf{a} = 3$  and  $\mathbf{b} = \mathbf{c}$ , the corresponding map expressed in bases is a  $3\mathbf{b} \times 3\mathbf{b}$  matrix. If  $a_0, a_1, a_2$  is a basis of  $A$  and one chooses bases of  $B, C$ , then elements of  $B \otimes C$  may be written as matrices, and  $T = a_0 \otimes X_0 + a_1 \otimes X_1 + a_2 \otimes X_2$ , where the  $X_j$  are size  $\mathbf{b}$  square matrices. With appropriate ordering, the corresponding matrix for  $T_A^{\wedge 1}$  is

$$Mat(T_A^{\wedge 1}) = \begin{pmatrix} 0 & X_2 & -X_1 \\ -X_2 & 0 & X_0 \\ X_1 & -X_0 & 0 \end{pmatrix}.$$

Now assume  $X_0$  is invertible and change bases such that it is the identity matrix. Recall the formula for block matrices

$$(2) \quad \det \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \det(W) \det(X - ZW^{-1}Y),$$

assuming  $W$  is invertible. Then, using the  $(\mathbf{b}, 2\mathbf{b})$  blocking (so  $X = 0$  in (2))

$$\det Mat(T_A^{\wedge 1}) = \det(X_1 X_2 - X_2 X_1) = \det([X_1, X_2]).$$

showing the equivalence of the two perspectives for the highest order Strassen equation, which is the only relevant one for this paper.

Strassen's equations were generalized in [5] and the generalization was used to prove new lower bounds for the border rank of matrix multiplication. Let  $\mathbf{a} = 2p + 1$ . Consider  $T \otimes Id_{\Lambda^p A} \in A \otimes B \otimes C \otimes \Lambda^p A \otimes \Lambda^p A^*$  and its skew-symmetrization to

$$(3) \quad T_A^{\wedge p} : \Lambda^p A \otimes B^* \rightarrow \Lambda^{p+1} A \otimes C.$$

By the choices of dimensions, this is a linear map between vector spaces of the same dimension  $\binom{2p+1}{p}\mathbf{b}$ . The result is that if the border rank of  $T$  is at most  $r$ , then the rank of the linear map  $T_A^{\wedge p}$  is at most  $r \binom{2p}{p}$ .

Writing  $T = a_0 \otimes X_0 + \cdots + a_{2p} \otimes X_{2p}$ , the expression of (3) in appropriate bases is of the form

$$(4) \quad \begin{pmatrix} 0 & Q \\ \tilde{Q} & R \end{pmatrix}$$

where this matrix is blocked  $((\binom{2p}{p+1}\mathbf{b}, \binom{2p}{p}\mathbf{b}) \times ((\binom{2p}{p+1}\mathbf{b}, \binom{2p}{p}\mathbf{b}))$ ,

$$R = \begin{pmatrix} & & X_0 \\ & \cdot & \\ X_0 & & \end{pmatrix},$$

$Q$  has entries in blocks consisting of  $X_1, \dots, X_{2p}$  and zero, and  $\tilde{Q}$  is the block transpose of  $Q$  except that if an index is even, the block is multiplied by  $-1$  (blocks with odd indices stay the same).

For example, when  $p = 2$ ,

$$Q = \begin{pmatrix} 0 & 0 & 0 & X_1 & -X_2 & X_3 \\ 0 & X_1 & -X_2 & 0 & 0 & X_3 \\ X_1 & 0 & -X_3 & 0 & X_4 & 0 \\ X_2 & -X_3 & 0 & X_4 & 0 & 0 \end{pmatrix}.$$

Note that if  $X_0$  is the identity matrix, the determinant of (4) when  $p = 2$  equals the determinant of

$$(5) \quad \begin{pmatrix} 0 & [X_1, X_2] & [X_1, X_3] & [X_1, X_4] \\ [X_2, X_1] & 0 & [X_2, X_3] & [X_2, X_4] \\ [X_3, X_1] & [X_3, X_2] & 0 & [X_3, X_4] \\ [X_4, X_1] & [X_4, X_2] & [X_4, X_3] & 0 \end{pmatrix}$$

In general, when  $X_0$  is the identity matrix, the determinant of (4) equals the determinant of the block  $2p\mathbf{b} \times 2p\mathbf{b}$  matrix whose  $(i, j)$ -th block entry is  $[X_i, X_j]$ .

*Remark 3.1.* It would be interesting to have direct geometric interpretations of the hypersurface in the Grassmannian  $G(2p, \mathfrak{sl}(B))$  of  $2p$ -dimensional subspaces of the traceless endomorphisms  $\mathfrak{sl}(B)$  that satisfy the equation that, if one takes a basis  $X_1, \dots, X_{2p}$  of the subspace, that the determinant of the matrix whose block entries are  $[X_i, X_j]$  is zero. The derivation shows that the vanishing of the determinant is independent of our choice of basis.

#### 4. PROOF OF THEOREM 1.2

Recall the following standard lemma (see, e.g., [4, §11.5]):

**Lemma 4.1.** *Let  $U$  be a vector space, let  $P$  be a polynomial on  $U$  of degree  $d$ . Let  $u_1, \dots, u_{\mathbf{n}}$  be a basis of  $U$ . Then there exists a subset  $u_{i_1}, \dots, u_{i_s}$  of cardinality  $s \leq d$  such that  $P|_{\langle u_{i_1}, \dots, u_{i_s} \rangle}$  is not identically zero.*

Lemma 4.1 says that a quadric surface in  $\mathbb{P}^3$  cannot contain three lines whose pairwise intersections span  $\mathbb{P}^3$ , and more generally, a hypersurface of degree  $d$  in  $\mathbb{P}U$  cannot contain the  $\binom{\mathbf{u}}{d}$   $\mathbb{P}^{d-1}$ 's of intersection of a collection of hyperplanes whose dual points span  $\mathbb{P}U^*$ .

Let  $G(k, A)$  denote the Grassmannian of  $k$ -planes through the origin in the vector space  $A$ . Given  $F \in G(\ell, A)$ , define the Schubert variety  $\Sigma_F^k := \{E \in G(k, A^*) \mid E \subset F^\perp\}$ .

**Lemma 4.2.** *Let  $b = (a_1, \dots, a_{\mathbf{n}^2})$  be a basis of  $A = N^* \otimes L$ . For each  $I \subset [\mathbf{n}^2]$  of cardinality  $4p+1$ , let  $F_I = \langle a_{i_1}, \dots, a_{i_{4p+1}} \rangle \in G(4p+1, A)$ . Let  $\Sigma_b = \cup_I \Sigma_{F_I}^{2p+1} \subset G(2p+1, A^*)$ .*

*Let  $M := Id_L \otimes Id_N \in A \otimes N \otimes L^*$ . Consider the polynomial  $Q$  on  $G(2p+1, A^*)$  given by, for  $A' \in G(2p+1, A^*)$ , the determinant of the linear map*

$$(M|_{A' \otimes L \otimes N^*})_{A'}^{\wedge p} : \Lambda^p A' \otimes L \rightarrow \Lambda^{p+1} A' \otimes N.$$

*Then for all bases  $b$ ,  $Q|_{\Sigma_b}$  is not identically zero.*

As described above, a choice of a general  $\alpha^0 \in A^*$  enables one to translated the vanishing of  $Q$  to the vanishing of a polynomial  $Q_0$  on  $G(2p, \mathfrak{sl}(L))$ .

To make the connection with Bläser's work more transparent, Lemma 4.2 may be rephrased as follows:

**Lemma 4.3.** *Let  $A = N^* \otimes L$ , where  $\mathbf{l} = \mathbf{n}$ . Given any basis of  $A$ , there exists a subset of at least  $\mathbf{n}^2 - (4p+1)\mathbf{n}$  basis vectors, and elements  $\alpha^0, \alpha^1, \dots, \alpha^{2p}$  of  $A^*$ , such that*

- (1)  $\alpha^0$  is of maximal rank, and thus may be used to identify  $L \simeq N$  and  $A$  as a space of endomorphisms. (I.e., in bases  $\alpha^0$  is the identity matrix.)
- (2) Choosing a basis of  $L$ , so the  $\alpha^j$  become  $\mathbf{n} \times \mathbf{n}$  matrices, the size  $2p\mathbf{n}$  block matrix whose  $(i, j)$ -th block is  $[\alpha^i, \alpha^j]$  has nonzero determinant, and
- (3) The subset of  $\mathbf{n}^2 - (4p+1)\mathbf{n}$  basis vectors annihilate  $\alpha^0, \alpha^1, \dots, \alpha^{2p}$ .

*Proof.* Assume a basis  $b$  has been given. By Lemma 4.1 with  $P = \det_{\mathbf{n}}$ , we may find a subset  $S_1$  of at most  $\mathbf{n}$  elements of our basis of  $A$  with some  $\alpha^0 \in \text{Span}(S_1)$  with  $\det_{\mathbf{n}}(\alpha^0) \neq 0$ . Use  $\alpha^0 : L \rightarrow N$  to identify  $N \simeq L$  which enables us to now consider  $A = \mathfrak{gl}(L)$  as an algebra with  $\alpha^0$  playing the role of the identity element.

Let  $v_{1,0}, \dots, v_{2p,0} \in A$  be linearly independent and not equal to any of the given basis vectors. Let  $E_0 = \langle v_{1,0}, \dots, v_{2p,0} \rangle$ . Work locally on an affine open subset  $\mathbb{A} \subset G(2p, \mathfrak{sl}(L)) = G(2p, A^* / \langle \alpha^0 \rangle)$  about  $E_0$ . Extend  $v_{1,0}, \dots, v_{2p,0}$  to a basis  $v_{1,0}, \dots, v_{2p,0}, w_1, \dots, w_{\mathbf{n}^2 - 2p - 1}$  of  $\mathfrak{sl}(L)$ . Take local coordinates  $(f_s^\mu)$ ,  $1 \leq s \leq 2p$ ,  $1 \leq \mu \neq \mathbf{n}^2 - 2p - 1$ , on  $\mathbb{A} \subset G(2p, \mathfrak{sl}(L))$  by writing  $v_s = v_{s,0} + f_s^\mu w_\mu$ . That is,  $E(f) \in \mathbb{A}$  is  $E(f_s^\mu) = \langle v_{1,0} + f_1^\mu w_\mu, \dots, v_{2p,0} + f_{2p}^\mu w_\mu \rangle$ .

By [5, §5],  $Q_0$  is not identically zero on  $G(2p, \mathfrak{sl}(L))$  and therefore it is not identically zero on  $\mathbb{A}$ . By the discussion in §3, up to scale, the polynomial  $Q_0$  is  $\det([v_s, v_t])$ . The entries of  $v_s$  are linear in the  $f_s^\mu$ , so the entries of the matrix  $([v_s, v_t])$  are quadratic in the  $f_s^\mu$ . Thus  $Q_0$  is a polynomial of degree  $4p\mathbf{n}$  in the  $f_s^\mu$ .

Applying Lemma 4.3 with  $P = Q_0$ , and recalling that we have already used up to  $\mathbf{n}$  basis elements to obtain  $\alpha^0$ , gives  $(4p+1)\mathbf{n}$  basis elements such that  $Q_0$  restricted to their span is not identically zero.  $\square$

*Remark 4.4.* The only property of  $Q_0$  that was used in the proof was its degree. It may be possible to improve the lower order terms in Theorem 1.1 by taking properties of  $Q_0$  into account. For example, Bläser's theorem is the case  $p = 1$ . There the polynomial

$$\det \begin{pmatrix} 0 & [\alpha^1, \alpha^2] \\ -[\alpha^1, \alpha^2] & 0 \end{pmatrix}$$

is a square, and one may work instead with  $\det([\alpha^1, \alpha^2])$ , which explains the improvement in the lower order terms in the case  $p = 1$ .

*Proof of Theorem 1.2.* Let  $\phi$  be an expression of  $M_{\langle \mathbf{n}, \mathbf{n}, \mathbf{m} \rangle}$  as a sum of  $r$  rank one tensors, and assume  $r$  to be minimal. Since  $\text{Lker}(M_{\langle \mathbf{n}, \mathbf{n}, \mathbf{m} \rangle}) = 0$  (i.e.,  $\forall \alpha \in A^* \setminus 0, \exists \beta \in B^*$  such that  $M(\alpha, \beta) \neq 0$ ) we may write  $\phi = \psi_1 + \psi_2$  with  $\mathbf{R}(\psi_1) = \mathbf{n}^2$ ,  $\mathbf{R}(\psi_2) = r - \mathbf{n}^2$  and  $\text{Lker}(\psi_1) = 0$ . Now consider the  $\mathbf{n}^2$  elements of  $A^*$  appearing in  $\psi_1$ . Since they span  $A^*$ , by Lemma 4.3 we may choose a subset of  $\mathbf{n}^2 - (4p+1)\mathbf{n}$  of them that annihilate a maximal rank element  $\alpha^0$  and some  $\alpha^1, \dots, \alpha^{2p}$  such that, choosing bases, the determinant of the matrix  $([\alpha^i, \alpha^j])$  is nonzero. Let  $\phi_1$  denote the sum of all monomials in  $\psi_1$  whose  $A^*$  terms annihilate  $\alpha^0, \alpha^1, \dots, \alpha^{2p}$ , so  $\mathbf{R}(\phi_1) \geq \mathbf{n}^2 - (4p+1)\mathbf{n}$ . Let  $\phi_2 = \psi_1 - \phi_1 + \psi_2$ .

Now  $\mathbf{R}(\phi_2) \geq \mathbf{n}\mathbf{m} \frac{2p+1}{p+1}$  because the determinant of the linear map  $M_{\langle \mathbf{n}, \mathbf{n}, \mathbf{m} \rangle}|_{A' \otimes B^* \otimes C^*} : \Lambda^p A' \otimes B^* \rightarrow \Lambda^{p+1} A' \otimes C$  is the determinant of the linear map  $M|_{A' \otimes L \otimes N^*}$  raised to the  $\mathbf{m}$ -th power, and the

determinant of the second map is nonzero by the argument in [5, §5]. Finally  $\mathbf{R}(M_{(\mathbf{n}, \mathbf{n}, \mathbf{m})}) = \mathbf{R}(\phi_1) + \mathbf{R}(\phi_2)$ .  $\square$

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